AN EQUALITY IN STOCHASTIC PROCESSES

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1. Introduction

The equality presented in this paper is extremely simple, but has proven useful on a number of occasions in deriving certain transition probabilities where other approaches such as the Laplace transform or the generating function become untidy (see, for example, [7]). The equality is not completely unknown; it has appeared, in a slightly different form, in a two dimensional process ([2], p. 102); and it is obvious for the simple Poisson process. The purpose of this paper is to state it in a general form and to demonstrate its validity and usefulness with a number of examples.

2. The equality

Let $\{X(t); t \in T\}$ be a time dependent Markov process defined over the interval T: $[0, \infty)$. For each $t \in T$, the random variable $X(t)$ assumes nonnegative integer values with the transition probability

$$P_{ik}(t_0, t) = Pr\{X(t) = k | X(t_0) = i\}, \qquad 0 \leqq t_0 \leqq t < \infty,\ i \leqq k;\ i, k = 0, 1, \cdots . \tag{1}$$

Our discussion is related only to nondecreasing processes where the value of $X(t)$ is increased by the occurrence of an event (for example, the pure birth process), or the nonincreasing processes (for example, the pure death process). The equality will be presented only for the former cases. However, an example of the pure death process will be given in Section 3.

For each i, we assume the existence of a continuous function $\lambda_i(\tau)$ such that

$$\frac{\partial}{\partial t} P_{ij}(\tau, t)\bigg|_{t=\tau} = \begin{cases} \lambda_i(\tau) & \text{for } j = i + 1, \\ -\lambda_i(\tau) & \text{for } j = i, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

It follows that the transition probabilities in (1) satisfy the forward differential equations

$$\begin{aligned} \frac{d}{dt} P_{ii}(t_0, t) &= -\lambda_i(t) P_{ii}(t_0, t), \\ \frac{d}{dt} P_{ik}(t_0, t) &= -\lambda_k(t) P_{ik}(t_0, t) + \lambda_{k-1}(t) P_{i,k-1}(t_0, t), \\ & \qquad i \leqq k;\ i, k = 0, 1, \cdots . \end{aligned} \tag{3}$$

The first equation in (3) has the solution

$$P_{ii}(t_0, t) = g_i(t_0, t) \tag{4}$$

where $g_i(t_0, t)$ denotes $\exp\left\{-\int_{t_0}^{t} \lambda_i(\xi)\, d\xi\right\}$, while the second equations give the recursive relationship

$$P_{ik}(t_0, t) = \int_{t_0}^{t} P_{i,k-1}(t_0, t_{k-1})\lambda_{k-1}(t_{k-1})P_{kk}(t_{k-1}, t)\, dt_{k-1}. \tag{5}$$

The probability $P_{i,k-1}(t_0, t_{k-1})$ may be written in the form of (5):

$$P_{i,k-1}(t_0, t_{k-1}) = \int_{t_0}^{t_{k-1}} P_{i,k-2}(t_0, t_{k-2})\lambda_{k-2}(t_{k-2})P_{k-1,k-1}(t_{k-2}, t_{k-1})\, dt_{k-2}. \tag{6}$$

Substituting (6) in (5) yields

$$P_{i,k}(t_0, t) = \int_{t_0}^{t} \int_{t_0}^{t_{k-1}} P_{i,k-2}(t, t_{k-2}) \prod_{\ell=k-2}^{k-1} \lambda_\ell(t_\ell)P_{\ell+1,\ell+1}(t_\ell, t_{\ell+1})\, dt_\ell, \tag{7}$$

or, upon integration with respect to t_{k-1},

$$P_{i,k}(t_0, t) = \int_{0}^{t} P_{i,k-2}(t_0, t_{k-2})\lambda_{k-2}(t_{k-2})P_{k-1,k}(t_{k-2}, t)\, dt_{k-2}. \tag{8}$$

Equation (8) indicates a sequence of transitions taking place in the interval $(t_0, t): i \to k - 2$ in (t_0, t_{k-2}), $k - 2 \to k - 1$ in $(t_{k-2}, t_{k-2} + dt_{k-2})$, and $k - 1 \to k$ in $(t_{k-2} + dt_{k-2}, t)$, for $t_0 \leqq t_{k-2} \leqq t$.

Equation (7) can be extended by repeated substitutions of (5) in (7) beginning with $k = k - 2$. Consequently, we arrive at a formula,

$$P_{ik}(t_0, t) = \int_{t_0}^{t} \int_{t_0}^{t_{k-1}} \cdots \int_{t_0}^{t_{k-i+1}} P_{ii}(t_0, t_i) \prod_{\ell=i}^{k-1} \lambda_\ell(t_\ell)P_{\ell+1,\ell+1}(t_\ell, t_{\ell+1})\, dt_\ell, \tag{9}$$

which shows the occurrence of each transition from ℓ to $\ell + 1$ in $(t_\ell, t_\ell + dt_\ell)$. Now the transitions can be regrouped in any meaningful way one wishes by integrating the right side of (9). When a particular transition, $j \to j + 1$, is of concern, we integrate the right side of (9) with respect to t_ℓ, for $i \leqq \ell < j$ and $j < \ell \leqq k$ to obtain *the equality. Let j be an arbitrary but fixed integer, for $i \leqq j < k$, then*

$$P_{ik}(t_0, t) = \int_{t_0}^{t} P_{ij}(t_0, \tau)\lambda_j(\tau)P_{j+1,k}(\tau, t)\, d\tau. \tag{10}$$

Clearly, (5) is a special case of the equality (10).

An equality for the continuous case also can be derived, where we will be dealing with the density function $f_{xz}(t_0, t)$ instead of the probability $P_{ik}(t_0, t)$.

A similarity between the present equality and the Chapman-Kolmogorov equation should be noted. The transition probability in (1) involves the parameter t and the values of the random variable $X(t)$ (or the states of the system). The Chapman-Kolmogorov equation,

$$P_{ik}(t_0, t) = \sum_j P_{ij}(t_0, \tau)P_{jk}(\tau, t), \tag{11}$$

is related to a fixed but arbitrary time τ and varying states j at τ; whereas in equality (10) a state j is fixed and the integral is taken over the values of τ. This, however, is the extent of the similarity. In equality (10), we *require* a transition from j to $j+1$ to take place in $(\tau, \tau + d\tau)$ for which a probability $\lambda_j(\tau)\, d\tau$ is included, while in the Chapman-Kolmogorov equation there is no such requirement. This additional factor nullifies a complete analogy between the two equalities.

3. Examples

3.1. *The Poisson process.* In the nonhomogeneous case, the intensity $\lambda_i(\tau) = \lambda(\tau)$ is a function of time; the transition probability is given by

$$(12)\qquad P_{ik}(t_0, t) = \frac{\exp\left\{-\int_{t_0}^t \lambda(\xi)\, d\xi\right\}\left[\int_{t_0}^t \lambda(\xi)\, d\xi\right]^{k-i}}{(k-i)!}, \qquad i \leqq k.$$

According to equality (10), we should have

$$(13)\quad \frac{g(t_0, t)}{(k-i)!}\left(\int_{t_0}^t \lambda(\xi)\, d\xi\right)^{k-i} = \int_{t_0}^t \left[\frac{g(t_0, \tau)}{(j-i)!}\left(\int_{t_0}^\tau \lambda(\xi)\, d\xi\right)^{j-i}\right]\lambda(\tau)\left[\frac{g(\tau, t)}{(k-j-1)!}\left(\int_\tau^t \lambda(\xi)\, d\xi\right)^{k-j-1}\right] d\tau.$$

where $g(t_0, t)$ denotes $\exp\left\{-\int_{t_0}^t \lambda(\xi)\, d\xi\right\}$ and $g(\tau, t)$ denotes $\exp\left\{-\int_\tau^t \lambda(\xi)\, d\xi\right\}$. To verify (13), we introduce a function

$$(14)\qquad \theta(\tau) = \frac{\int_{t_0}^\tau \lambda(\xi)\, d\xi}{\int_{t_0}^t \lambda(\xi)\, d\xi}, \qquad d\theta(\tau) = \frac{\lambda(\tau)\, d\tau}{\int_{t_0}^t \lambda(\xi)\, d\xi},$$

and rewrite the right side of (13) as

$$(15)\quad \frac{g(t_0, t)}{(j-i)!(k-j-1)!}\left(\int_{t_0}^t \lambda(\xi)\, d\xi\right)^{k-i}\int_0^1 [\theta(\tau)]^{j-i}[1-\theta(\tau)]^{k-j-1}\, d\theta(\tau) = \frac{g(t_0, t)}{(k-i)!}\left(\int_{t_0}^t \lambda(\xi)\, d\xi\right)^{k-i},$$

and recover the left side of (13).

When $\lambda(\tau) = \lambda$, and $t_0 = 0$, (13) becomes

$$(16)\qquad \frac{e^{-\lambda t}(\lambda t)^{k-i}}{(k-i)!} = \int_0^t \frac{e^{-\lambda\tau}(\lambda\tau)^{j-i}}{(j-i)!}\lambda\frac{e^{-\lambda(t-\tau)}[\lambda(t-\tau)]^{k-j-1}}{(k-j-1)!}\, d\tau.$$

3.2. *The Yule process.* When the intensity $\lambda_i(\tau) = i\lambda(\tau)$, the differential equations in (3) define the Yule process,

$$(17)\qquad P_{ik}(t_0, t) = \frac{(k-1)!}{(k-i)!(i-1)!}(g(t_0, t))^i(1 - g(t_0, t))^{k-i}.$$

To verify equality (10), we need to show that

$$(18)\quad \frac{(k-1)!}{(k-i)!(i-1)!}(g(t_0,t))^i(1-g(t_0,t))^{k-i}$$
$$= \int_{t_0}^{t} \frac{(j-1)!}{(j-i)!(i-1)!}(g(t_0,\tau))^i(1-g(t_0,\tau))^{j-i}j\lambda(\tau)$$
$$\frac{(k-1)!}{(k-j-1)!j!}(g(\tau,t))^{j+1}(1-g(\tau,t))^{k-j-1}\,d\tau,$$

where $g(t_0,\tau)$ denotes $\exp\left\{-\int_{t_0}^{\tau}\lambda(\xi)\,d\xi\right\}$. Let

$$(19)\qquad \theta(\tau) = \frac{1-g(\tau,t)}{1-g(t_0,t)},$$

so that

$$(20)\qquad d\theta(\tau) = \frac{-g(\tau,t)\lambda(\tau)\,d\tau}{1-g(t_0,t)} \quad\text{and}\quad 1-\theta(\tau) = \frac{g(\tau,t)-g(t_0,t)}{1-g(t_0,t)}.$$

Substituting (19) and (20) in the right side of (18) and simplifying yields

$$(21)\quad \frac{(k-1)!}{(j-i)!(i-1)!(k-j-1)!}(g(t_0,t))^i(1-g(t_0,t))^{k-i}$$
$$\int_0^1 [1-\theta(\tau)]^{j-i}[\theta(\tau)]^{k-j-1}\,d\theta(\tau)$$
$$= \frac{(k-1)!}{(k-i)!(i-1)!}(g(t_0,t))^i(1-g(t_0,t))^{k-i},$$

which is equal to the left side of (18), proving (18).

3.3. *The pure birth process.* To verify equality (10) for the present case, we need the following lemma.

LEMMA 1. *Whatever may be distinct real numbers,* $\lambda_1, \lambda_2, \cdots, \lambda_n$,

$$(22)\qquad \frac{1}{\prod_{\gamma=2}^{n}(\lambda_1-\lambda_\gamma)} + \cdots + \frac{1}{\prod_{\gamma=1}^{n-1}(\lambda_n-\lambda_\gamma)} = 0.$$

Several proofs of the lemma are given in Chiang [2], p. 126; we do not present the details here.

The pure birth process is a different form from the Yule process in that the intensity function $\lambda_i(\tau) = \lambda_i$ is a function of i, but is assumed to be independent of time τ. When λ_i are distinct for different i, then the differential equation (3) has the solution (see Feller [5], p. 449, and [2], p. 51),

$$(23)\qquad P_{ik}(t_0,t) = (-1)^{k-i}\lambda_i\lambda_{i+1}\cdots\lambda_{k-1}\sum_{\alpha=i}^{k}\frac{\exp\{-\lambda_\alpha(t-t_0)\}}{\prod_{\gamma=i,\gamma\neq\alpha}^{k}(\lambda_\alpha-\lambda_\gamma)}.$$

Similarly,

$$(24)\qquad P_{ij}(t_0,\tau) = (-1)^{j-i}\lambda_i\lambda_{i+1}\cdots\lambda_{j-1}\sum_{\alpha=i}^{j}\frac{\exp\{-\lambda_\alpha(\tau-t_0)\}}{\prod_{\gamma=i,\gamma\neq\alpha}^{j}(\lambda_\alpha-\lambda_\gamma)}$$

and

$$(25)\quad P_{j+1,k}(\tau, t) = (-1)^{k-j-1}\lambda_{j+1}\lambda_{j+2}\cdots\lambda_{k-1}\sum_{\beta=j+1}^{k}\frac{\exp\{-\lambda_\beta(t-\tau)\}}{\prod_{\delta=j+1,\delta\neq\beta}^{k}(\lambda_\beta-\lambda_\delta)}.$$

When (24) and (25) are substituted in (10), we find on the right side of (10),

$$(26)\quad \int_{t_0}^{t} P_{ij}(t_0, \tau)\lambda_j P_{j+1,k}(\tau, t)\, d\tau$$

$$= (-1)^{k-i-1}\lambda_i\lambda_{i+1}\cdots\lambda_{k-1}\sum_{\alpha=i}^{j}\sum_{\beta=j+1}^{k}\frac{\int_{t_0}^{t}\exp\{-\lambda_\alpha(\tau-t_0)\}\exp\{-\lambda_\beta(t-\tau)\}\,d\tau}{\prod_{\gamma=i,\gamma\neq\alpha}^{j}(\lambda_\alpha-\lambda_\gamma)\prod_{\delta=j+1,\delta\neq\beta}^{k}(\lambda_\beta-\lambda_\delta)},$$

where

$$(27)\quad \int_{t_0}^{t}\exp\{-\lambda_\alpha(\tau-t_0)\}\exp\{-\lambda_\beta(t-\tau)\}\,d\tau$$

$$= -\left[\frac{\exp\{-\lambda_\alpha(t-t_0)\}}{\lambda_\alpha-\lambda_\beta}+\frac{\exp\{-\lambda_\beta(t-t_0)\}}{\lambda_\beta-\lambda_\alpha}\right].$$

Therefore, (26) may be rewritten

(28)

$$(-1)^{k-i}\lambda_i\lambda_{i+1}\cdots\lambda_{k-1}\left[\sum_{\alpha=i}^{j}\sum_{\beta=j+1}^{k}\frac{\exp\{-\lambda_\alpha(t-t_0)\}}{(\lambda_\alpha-\lambda_\beta)\prod_{\gamma=i,\gamma\neq\alpha}^{j}(\lambda_\alpha-\lambda_\gamma)\prod_{\delta=j+1,\delta\neq\beta}^{k}(\lambda_\beta-\lambda_\delta)}\right.$$

$$\left.+\sum_{\alpha=i}^{j}\sum_{\beta=j+1}^{k}\frac{\exp\{-\lambda_\beta(t-t_0)\}}{(\lambda_\beta-\lambda_\alpha)\prod_{\gamma=i,\gamma\neq\alpha}^{j}(\lambda_\alpha-\lambda_\gamma)\prod_{\delta=j+1,\delta\neq\beta}^{k}(\lambda_\beta-\lambda_\delta)}\right].$$

In the first term inside the brackets, for each α, we compute the sum

$$(29)\quad \left\{\sum_{\beta=j+1}^{k}\frac{1}{(\lambda_\alpha-\lambda_\beta)\prod_{\delta=j+1,\delta\neq\beta}^{k}(\lambda_\beta-\lambda_\delta)}\right\}\frac{\exp\{-\lambda_\alpha(t-t_0)\}}{\prod_{\gamma=i,\gamma\neq\alpha}^{j}(\lambda_\alpha-\lambda_\gamma)}$$

$$= \left\{\frac{1}{\prod_{\delta=j+1}^{k}(\lambda_\alpha-\lambda_\delta)}\right\}\frac{\exp\{-\lambda_\alpha(t-t_0)\}}{\prod_{\gamma=i,\gamma\neq\alpha}^{j}(\lambda_\alpha-\lambda_\gamma)} = \frac{\exp\{-\lambda_\alpha(t-t_0)\}}{\prod_{\gamma=i,\gamma\neq\alpha}^{k}(\lambda_\alpha-\lambda_\gamma)},$$

since, according to Lemma 1, for any distinct numbers $\lambda_\alpha, \lambda_{j+1}, \lambda_{j+2}, \cdots, \lambda_k$,

$$(30)\quad \frac{1}{(\lambda_\alpha-\lambda_{j+1})\prod_{\gamma=j+2}^{k}(\lambda_{j+1}-\lambda_\gamma)}+\cdots+\frac{1}{(\lambda_\alpha-\lambda_k)\prod_{\gamma=j+1}^{k-1}(\lambda_k-\lambda_\gamma)}$$

$$= \frac{1}{\prod_{\gamma=j+1}^{k}(\lambda_\alpha-\lambda_\gamma)}.$$

Similarly, in the second term inside the brackets of (28) we find for each β,

(31)

$$\left\{\sum_{\alpha=i}^{j} \frac{1}{(\lambda_\beta - \lambda_\alpha) \prod_{\gamma=i,\gamma\neq\alpha}^{j} (\lambda_\alpha - \lambda_\gamma)}\right\} \frac{\exp\{-\lambda_\beta(t - t_0)\}}{\prod_{\delta=j+1,\delta\neq\beta}^{k} (\lambda_\beta - \lambda_\delta)} = \frac{\exp\{-\lambda_\beta(t - t_0)\}}{\prod_{\delta=i,\delta\neq\beta}^{k} (\lambda_\beta - \lambda_\delta)}.$$

Substituting (29) and (31) in (28) yields

$$(32) \qquad (-1)^{k-i}\lambda_i\lambda_{i+1} \cdots \lambda_{k-1} \sum_{\alpha=i}^{k} \frac{\exp\{-\lambda_\alpha(t - t_0)\}}{\prod_{\gamma=i,\gamma\neq\alpha}^{k} (\lambda_\alpha - \lambda_\gamma)},$$

which is equal to (23), as was to be shown.

3.4. *The Pólya process.* The Pólya process is determined by the differential equation (3) with the intensity function

$$(33) \qquad \lambda_k(t) = \frac{\lambda + \lambda a k}{1 + \lambda a t},$$

where both λ and a are nonnegative constants. Solving (3) for $P_{ik}(t_0, t)$, we have

$$(34) \quad P_{ik}(t_0, t) = \frac{\Gamma(k + 1/a)}{(k - i)!\Gamma(i + 1/a)} \left(\frac{1 + \lambda a t_0}{1 + \lambda a t}\right)^{i+1/a} \left(1 - \frac{1 + \lambda a t_0}{1 + \lambda a t}\right)^{k-i}.$$

Using this general formula for the probabilities $P_{ij}(t_0, \tau)$ and $P_{j+1,k}(\tau, t)$ in (10), we obtain

$$(35) \quad \int_{t_0}^{t} P_{ij}(t_0, \tau)\lambda_j(\tau)P_{j+1,k}(\tau, t)\, d\tau$$

$$= \frac{\Gamma(j + 1/a)}{(j - i)!\Gamma(i + 1/a)} \frac{\Gamma(k + 1/a)}{(k - j - 1)!\Gamma(j + 1 + 1/a)} \int_{t_0}^{t} \left(\frac{1 + \lambda a t_0}{1 + \lambda a \tau}\right)^{i+1/a}$$

$$\left(1 - \frac{1 + \lambda a t_0}{1 + \lambda a \tau}\right)^{j-i} \left(\frac{\lambda + \lambda a j}{1 + \lambda a \tau}\right)\left(\frac{1 + \lambda a \tau}{1 + \lambda a t}\right)^{j+1+1/a} \left(1 - \frac{1 + \lambda a \tau}{1 + \lambda a t}\right)^{k-j-1} d\tau,$$

where the integral is simply

(36)

$$(1 + \lambda a t_0)^{i+1/a}(1 + \lambda a t)^{-(k+1/a)}(\lambda a)^{k-i-1}(\lambda + \lambda a j) \int_{t_0}^{t} (\tau - t_0)^{j-i}(t - \tau)^{k-j-1}\, d\tau$$

$$= (1 + \lambda a t_0)^{i+1/a}(1 + \lambda a t)^{-(k+1/a)}(\lambda a)^{k-i-1}$$

$$(\lambda + \lambda a j)(t - t_0)^{k-i} \frac{(j - i)!(k - j - 1)!}{(k - i)!}$$

$$= \left(\frac{1 + \lambda a t_0}{1 + \lambda a t}\right)^{i+1/a} \left(1 - \frac{\lambda a t_0}{1 + \lambda a t}\right)^{k-i} \left(j + \frac{1}{a}\right) \frac{(j - i)!(k - j - 1)!.}{(k - i)!}$$

When the last expression in (36) is substituted for the integral in (35), the right side member of (35) becomes identical to that in (34), proving equality (10) for the Pólya process.

3.5. *The pure death process.* The pure death process is different from all the preceding processes; in this case the population size is decreasing instead of increasing with time t. The transition probability $P_{ik}(t_0, t)$ holds for $k \leqq i$ and satisfies the differential equation

$$\frac{d}{dt} P_{ik}(t_0, t) = -k\mu(t)P_{ik}(t_0, t) + (k+1)\mu(t)P_{i,k+1}(t_0, t), \qquad k = 0, 1, \cdots, i, \tag{37}$$

where the intensity function $\mu(t)$ is also known as the force of mortality. The solution of (37) is

$$P_{ik}(t_0, t) = \frac{i!}{k!(i-k)!} (h(t_0, t))^k (1 - h(t_0, t))^{i-k}, \tag{38}$$

where $h(t_0, t)$ denotes $\exp\left\{-\int_{t_0}^t \mu(\xi)\, d\xi\right\}$. The equality for the death process assumes a form slightly different from (10);

$$P_{ik}(t_0, t) = \int_{t_0}^t P_{ij}(t_0, \tau) j\mu(\tau) P_{j-1,k}(\tau, t)\, d\tau, \qquad 0 \leqq k < j \leqq i. \tag{39}$$

Verification of (39) is straightforward. Using (37) we can write

$$\begin{aligned} &\int_{t_0}^t P_{ij}(t_0, \tau) j\mu(\tau) P_{j-1,k}(\tau, t)\, d\tau \\ &= \int_{t_0}^t \frac{i!}{j!(i-j)!} (h(t_0, \tau))^j (1 - h(t_0, \tau))^{i-j} j\mu(\tau) \\ &\qquad \frac{(j-1)!}{k!(j-1-k)!} (h(\tau, t))^k (1 - h(\tau, t))^{j-1-k}\, d\tau, \end{aligned} \tag{40}$$

where $h(\tau, t)$ denotes $\exp\left\{-\int_\tau^t \mu(\xi)\, d\xi\right\}$ and $h(t_0, \tau)$ denotes $\exp\left\{-\int_{t_0}^\tau \mu(\xi)\, d\xi\right\}$. Now let

$$\theta(\tau) = \frac{1 - h(t_0, \tau)}{1 - h(t_0, t)}, \qquad d\theta(\tau) = \frac{h(t_0, \tau)\mu(\tau)\, d\tau}{1 - h(t_0, t)}. \tag{41}$$

Then (40) may be rewritten as

$$\begin{aligned} &\frac{i!}{(i-j)!k!(j-1-k)!} (h(t_0, t))^k (1 - h(t_0, t))^{i-k} \int_0^1 (\theta(\tau))^{i-j} (1 - \theta(\tau))^{j-1-k}\, d\theta(\tau) \\ &= \frac{i!}{k!(i-k)!} (h(t_0, t))^k (1 - h(t_0, t))^{i-k} \\ &= P_{ik}(t_0, t), \end{aligned} \tag{42}$$

which proves the equality in (39).

4. An application

Theoretical significance of equality (10) is yet to be assessed, depending to a great extent on its relevance in developing stochastic processes. The following example illustrates its usefulness as a means of deriving certain transition probabilities.

The example in question is the so-called "simple stochastic epidemic," which has been extensively studied by N. T. J. Bailey [1]. According to this model, a population consists of two groups of individuals: infectives and susceptibles. At the initial time t_0, there are one infective and N susceptibles. For each t, for $t \in [t_0, \infty)$, there are $X(t)$ infectives and $N + 1 - X(t)$ susceptibles. The probability distribution of the random variable $X(t)$ satisfies the system of differential equations

$$\frac{d}{dt} P_{1k}(t_0, t) = -a_k\beta(t)P_{1k}(t_0, t) + a_{k-1}\beta(t)P_{1,k-1}(t_0, t), \qquad k = 1, \cdots, N + 1, \tag{43}$$

where

$$a_k = k(N + 1 - k) \qquad k = 1, \cdots, N; \; a_0 = a_{N+1} = 0, \tag{44}$$

with the initial condition $P_{11}(t_0, t_0) = 1$, and the constant $\beta(t)$ is known as the infection rate. The Laplace transform and the probability generating function have been used to solve the differential equations (43). Because the coefficient a_k is a quadratic function of k, for each k, the computations involved are quite complex. The partial differential equation for the probability generating function, for example, is of the second order. However, when equality (10) is used, one can write down the solution almost immediately. The complete solution and the detailed discussion on related points have been given in [7], and will not be repeated here.

5. The equalities in a two dimensional process

Equality (10) can be extended to multidimensional processes, where the random variables concerned are the number of transitions rather than population sizes. To be specific, we use a two dimensional process for illustration. In the birth, death and other processes discussed in Section 3, there is only one transient state for each individual: the state of "living." In a two dimensional process, each individual may be in either one of two transient states, S_1 and S_2. State S_1 may be interpreted as the healthy state, and S_2 the illness state. In addition, there is an absorbing state R, the death state. A person is in state S_1 if he is well, in S_2 if he is ill. A transition from S_1 to S_2 means the occurrence of an illness or a relapse, while a transition from S_2 to S_1 means recovery. When a person dies, he enters the death state R from either S_1 or S_2 depending upon whether he is in state S_1 or S_2 at the time of death. Fix and Neyman have discussed extensively this model in their study of the probabilities of relapse, recovery, and death for cancer patients [6] (see also Du Pasquier [4]).

During a time interval (t_0, t) an individual may leave one transient state for another. We are interested in the number of transitions that he makes between the two transient states and the corresponding probabilities, which are defined as follows:

$$
\begin{aligned}
(45)\quad P_{\alpha\beta}^{(m)}(t_0, t) &= Pr\ \{\text{an individual in state } S_\alpha \text{ at time } t_0 \text{ will leave } S_\alpha\ m \text{ times} \\
&\qquad \text{during } (t_0, t) \text{ and will be in } S_\beta \text{ at time } t\} \\
&= Pr\{M_{\alpha\beta}(t_0, t) = m\}, \qquad \alpha, \beta = 1, 2; m = 0, 1, \cdots .
\end{aligned}
$$

The random variable $M_{\alpha\beta}(t_0, t)$, corresponding to the probability $P_{\alpha\beta}^{(m)}(t_0, t)$, is thus the number of times the individual leaves S_α for S_β before he reaches S_β at time t. The sums

$$
(46)\qquad
\begin{aligned}
&\sum_{m=0}^{\infty} P_{\alpha\alpha}^{(m)}(t_0, t) = P_{\alpha\alpha}(t_0, t), \\
&\sum_{m=1}^{\infty} P_{\alpha\beta}^{(m)}(t_0, t) = P_{\alpha\beta}(t_0, t), \qquad \alpha \neq \beta; \alpha, \beta = 1, 2,
\end{aligned}
$$

are the probabilities that the individual will be at time t in S_α and S_β, respectively, regardless of the number of transitions he makes between t_0 and t. We assume that, for each τ, for $t_0 \leqq \tau \leqq t$, the derivatives

$$
(47)\qquad \frac{\partial}{\partial t} P_{\alpha\alpha}(\tau, t)\Big|_{t=\tau} = \nu_{\alpha\alpha}, \qquad \frac{\partial}{\partial t} P_{\alpha\beta}(\tau, t)\Big|_{t=\tau} = \nu_{\alpha\beta}
$$

exist and are independent of time τ, so that $\nu_{\alpha\alpha}$ is a negative constant and $\nu_{\alpha\beta}$ a positive constant. Explicit formulas of the probabilities $P_{\alpha\beta}^{(m)}(t_0, t)$ have been derived in terms of $\nu_{\alpha\alpha}$ and $\nu_{\alpha\beta}$ (see [2], Chapter 5).

Corresponding to equality (10) there are four equalities in the two dimensional process. They are

$$
(48)\qquad P_{\alpha\beta}^{(m)}(t_0, t) = \int_{t_0}^{t} P_{\alpha\alpha}^{(j)}(t_0, \tau)\nu_{\alpha\beta}P_{\beta\beta}^{(m-j-1)}(\tau, t)\, d\tau,
$$

$$
(49)\qquad P_{\alpha\beta}^{(m)}(t_0, t) = \int_{t_0}^{t} P_{\alpha\beta}^{(j)}(t_0, \tau)\nu_{\beta\alpha}P_{\alpha\beta}^{(m-j)}(\tau, t)\, d\tau,
$$

$$
(50)\qquad P_{\alpha\alpha}^{(m)}(t_0, t) = \int_{t_0}^{t} P_{\alpha\beta}^{(j)}(t_0, \tau)\nu_{\beta\alpha}P_{\alpha\alpha}^{(m-j)}(\tau, t)\, d\tau,
$$

and

$$
(51)\qquad P_{\alpha\alpha}^{(m)}(t_0, t) = \int_{t_0}^{t} P_{\alpha\alpha}^{(j)}(t_0, \tau)\nu_{\alpha\beta}P_{\beta\alpha}^{(m-j)}(\tau, t)\, d\tau,
$$

for any fixed but arbitrary j, for j between 0 and m. Equality (48), for example, holds for $j = 0, \cdots, m - 1$; while (49) holds for $j = 1, \cdots, m - 1$. These four equalities can be verified in a similar manner as equality (10). For the verification of the equality in (48), consider that an individual in state S_α at time t_0 will leave S_α m times and be in S_β at time t, and let j be a fixed number with $0 \leqq j < m$. Let the $(j + 1)$th exit transition from S_α to S_β take place in

$(\tau, \tau + d\tau)$, so that at τ he is in state S_α, the remaining $m - j - 1$ transitions from S_β occurring during (τ, t); the probability of this sequence of events is

$$P_{\alpha\alpha}^{(j)}(t_0, \tau)[\nu_{\alpha\beta}\, d\tau]P_{\beta\beta}^{(m-j-1)}(\tau, t). \tag{52}$$

Integrating (52) from $\tau = t_0$ to $\tau = t$ gives the equality in (48).

It has been shown in [2], pp. 102 to 104, that the multiple transition probabilities $P_{\alpha\beta}^{(m)}(t_0, t)$ and $P_{\alpha\alpha}^{(m)}(t_0, t)$ satisfy the equalities (48) through (51) for every m. Two simple cases are given below.

It is obvious that for $m = 0$,

$$P_{\alpha\alpha}^{(0)}(t_0, t) = \exp\{\nu_{\alpha\alpha}(t - t_0)\}, \qquad P_{\beta\beta}^{(0)}(t_0, t) = \exp\{\nu_{\beta\beta}(t - t_0)\}. \tag{53}$$

For $m = 1$, $P_{\alpha\beta}^{(1)}(t_0, t)$ is the probability that exactly one transition from S_α to S_β occurs in (t_0, t), and $P_{\alpha\alpha}^{(1)}(t_0, t)$ is the first return probability to the original state S_α at t after having left S_α once. The first passage probability is given by

$$P_{\alpha\beta}^{(1)}(t_0, t) = \frac{\nu_{\alpha\beta}}{\nu_{\alpha\alpha} - \nu_{\beta\beta}}(\exp\{\nu_{\alpha\alpha}(t - t_0)\} - \exp\{\nu_{\beta\beta}(t - t_0)\}), \qquad \alpha \neq \beta. \tag{54}$$

According to equality (48), we have

$$P_{\alpha\beta}^{(1)}(t_0, t) = \int_{t_0}^{t} P_{\alpha\alpha}^{(0)}(t_0, \tau)\nu_{\alpha\beta}P_{\beta\beta}^{(0)}(\tau, t)\, d\tau, \tag{55}$$

or, substituting (53) and (54) in (55),

$$\begin{aligned} \frac{\nu_{\alpha\beta}}{\nu_{\alpha\alpha} - \nu_{\beta\beta}}&(\exp\{\nu_{\alpha\alpha}(t - t_0)\} - \exp\{\nu_{\beta\beta}(t - t_0)\}) \\ &= \int_{t_0}^{t} \exp\{\nu_{\alpha\alpha}(\tau - t_0)\}\nu_{\alpha\beta}\exp\{\nu_{\beta\beta}(t - \tau)\}\, d\tau, \end{aligned} \tag{56}$$

which is easily shown to be true.

Equalities (48) and (50) (or (49) and (51)) can also be used to derive the general formulas for the probabilities $P_{\alpha\alpha}^{(m)}(t_0, t)$ and $P_{\alpha\beta}^{(m)}(t_0, t)$. The probability $P_{\alpha\beta}^{(1)}(t_0, t)$, if it is unknown, can be obtained from (55). Using the known probability $P_{\alpha\beta}^{(1)}(t_0, t)$, we can derive the first return probability $P_{\alpha\alpha}^{(1)}(t_0, t)$ from equality (50) for $m = 1, j = 1$,

$$P_{\alpha\alpha}^{(1)}(t_0, t) = \int_{t_0}^{t} P_{\alpha\beta}^{(1)}(t_0, \tau)\nu_{\beta\alpha}P_{\alpha\alpha}^{(0)}(\tau, t)\, d\tau. \tag{57}$$

Substituting (56) for $P_{\alpha\beta}^{(1)}(t_0, \tau)$ and (53) for $P_{\alpha\alpha}^{(0)}(\tau, t)$ in (57) and integrating the resulting expression yields

$$\begin{aligned} P_{\alpha\alpha}^{(1)}(t_0, t) = \frac{\nu_{\alpha\beta}\nu_{\beta\alpha}}{(\nu_{\alpha\alpha} - \nu_{\beta\beta})^2}&[(\nu_{\alpha\alpha} - \nu_{\beta\beta})(t - t_0)\exp\{\nu_{\alpha\alpha}(t - t_0)\} \\ &- (\exp\{\nu_{\alpha\alpha}(t - t_0)\} - \exp\{\nu_{\beta\beta}(t - t_0)\})]. \end{aligned} \tag{58}$$

Now using equalities (48) and (50) successively for $m = 2$, we obtain the second passage probability

(59)

$$P_{\alpha\beta}^{(2)}(t_0, t) = \frac{\nu_{\alpha\beta}^2 \nu_{\beta\alpha}}{(\nu_{\alpha\alpha} - \nu_{\beta\beta})^3} [(\nu_{\alpha\alpha} - \nu_{\beta\beta})(t - t_0)(\exp\{\nu_{\alpha\alpha}(t - t_0)\} + \exp\{\nu_{\beta\beta}(t - t_0)\}) - 2(\exp\{\nu_{\alpha\alpha}(t - t_0)\} - \exp\{\nu_{\beta\beta}(t - t_0)\})]$$

and the second return probability

$$(60) \quad P_{\alpha\alpha}^{(2)}(t_0, t) = \frac{\nu_{\alpha\beta}^2 \nu_{\beta\alpha}^2}{(\nu_{\alpha\alpha} - \nu_{\beta\beta})^4} [\tfrac{1}{2}(\nu_{\alpha\alpha} - \nu_{\beta\beta})^2 (t - t_0)^2 \exp\{\nu_{\alpha\alpha}(t - t_0)\} - (\nu_{\alpha\alpha} - \nu_{\beta\beta})(t - t_0)(2\exp\{\nu_{\alpha\alpha}(t - t_0)\} + \exp\{\nu_{\beta\beta}(t - t_0)\}) + 3(\exp\{\nu_{\alpha\alpha}(t - t_0)\} - \exp\{\nu_{\beta\beta}(t - t_0)\})].$$

The probabilities $P_{\alpha\beta}^{(m)}(t_0, t)$ and $P_{\alpha\alpha}^{(m)}(t_0, t)$ can all be successively derived in the same manner beginning with $m = 3$. It is interesting to note that, using this approach, the multiple transition probabilities can be derived even when the intensity functions $\nu_{\alpha\alpha}(t)$ and $\nu_{\alpha\beta}(t)$ are functions of time.

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